

ON THE ALGEBRAICITY OF PUISEUX SERIES.

MICHEL HICKEL AND MICKAËL MATUSINSKI

ABSTRACT. We deal with the algebraicity of a Puiseux series in terms of the properties of its coefficients. We show that the algebraicity of a Puiseux series for given bounded degree is determined by a finite number of explicit polynomial formulae. Conversely, given a vanishing polynomial, there is a closed-form formula for the coefficients of the series in terms of the coefficients of the polynomial and of an initial part of the series.

1. INTRODUCTION.

Let K be a zero characteristic field and \overline{K} its algebraic closure. We consider $K[[x]]$, the domain of formal power series with coefficients in K , and its fraction field $K((x))$. We denote by $K((\hat{x})) := \bigcup_{n=1}^{\infty} K((x^{1/n}))$ the field of formal Puiseux series (with coefficients in K). By the Newton-Puiseux theorem (see e.g. [Wal78, Theorem 3.1] and [RvdD84, Proposition p.314]), an algebraic closure of $K((x))$ is given by $\mathcal{P}_K := \bigcup_L L((\hat{x}))$ where L ranges over the finite extensions of K in \overline{K} . In particular, if $K = \overline{K}$, then $\mathcal{P}_K = K((\hat{x}))$. Among Puiseux series, we are interested in algebraic ones, say the Puiseux series which verify a polynomial equation with coefficients that are themselves polynomials in x : $P(x, y) \in K[x][y]$.

Among the numerous works concerning algebraic Puiseux series [vdP93, FS09, BD13], we deal with the following questions:

- **Reconstruction of a vanishing polynomial for a given algebraic Puiseux series.** Generically, a vanishing polynomial of a given algebraic power series can be computed as a Padé-Hermite approximant [BCG⁺14, Chap. 7]. In fact, the algebraicity of a Puiseux series can be encoded by the vanishing of certain determinants derived from the coefficients of the series. We extend this approach by showing how to reconstruct the coefficients of a vanishing polynomial by means of some minors of these determinants (see Section 3). More precisely, we show that, for a given bounded degree, there are finitely many universal polynomials allowing to check the algebraicity of a series and to perform this reconstruction (see Theorem 3.5).
- **Description of the coefficients of an algebraic Puiseux series in terms of the coefficients of a vanishing polynomial.** An approach consists in considering that the series coefficients verify a linear recurrence relation, which allows an asymptotic computation of the coefficients. This property follows classically from the fact that an algebraic Puiseux series is *differentially finite* (*D-finite*), that is, it satisfies a linear differential equation with coefficients in $K[x]$ [Com64, Sta78, Sta99, Sin80, CC86, CC87, BCS⁺07].

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Another approach consists in determining a closed-form expression in terms of the coefficients of a vanishing polynomial. In this direction, P. Flajolet and M. Soria (see the habilitation thesis of M. Soria (1990) and [FS]) proposed a formula in the case of a series satisfying a reduced Henselian equation (see the Definition 2.2 for this terminology) with complex coefficients. This formula extends to coefficients in an arbitrary zero characteristic field K via a work of Henrici [Hen64].

Here we complete this approach to the case of a Puiseux series which satisfies a general polynomial equation $P(x, y) = 0$, by showing that the coefficients of such series can be computed applying the Flajolet-Soria formula to a polynomial naturally derived from P (see Section 4).

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2. PRELIMINARIES

Let us denote $\mathbb{N} := \mathbb{Z}_{\geq 0}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\} = \mathbb{Z}_{>0}$. For any set \mathcal{E} , we will write $|\mathcal{E}| := \text{Card}(\mathcal{E})$. For any vector of natural numbers $K = (k_1, \dots, k_n)$, we set $K! := \prod_{i=1}^n k_i!$,

$|K| := \sum_{i=1}^n k_i$ and $\|K\| := \sum_{i=1}^n i k_i$. The floor function will be written $\lfloor q \rfloor$ for $q \in \mathbb{Q}$.

Let $\tilde{y}_0 = \sum_{n \geq n_0} \tilde{c}_n x^{n/p} \in K((\hat{x}))$, $\tilde{c}_{n_0} \neq 0$, a Puiseux series. We denote

$$\tilde{y}_0 = x^{(n_0-1)/p} \sum_{n \geq 1} c_n x^{n/p} = x^{(n_0-1)/p} \tilde{z}_0 \text{ with } c_1 \neq 0.$$

The series \tilde{y}_0 is a root of a polynomial $\tilde{P}(x, y) = \sum_{i,j} \tilde{a}_{i,j} x^i y^j$ of degree d_y in y if and only if the series $y_0 = \sum_{n \geq 1} c_n x^n$ is a root of $x^m \tilde{P}(x^p, x^{n_0-1} y)$, the latter being a polynomial for $m = \max\{0; (1 - n_0)d_y\}$. The existence of a polynomial \tilde{P} cancelling \tilde{y}_0 is equivalent to the one of a polynomial $P(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ cancelling y_0 , such that, for (i, j) belonging to the support of P , one has $i \equiv (n_0 - 1)j \pmod{p}$ if $n_0 - 1 \geq 0$, and $i \equiv (1 - n_0)(d_y - j) \pmod{p}$ otherwise. Thus the algebraicity of \tilde{y}_0 is equivalent to that of y_0 but *with constraints*. This leads us to the following definition:

Definition 2.1. Let \mathcal{F} and \mathcal{G} be two strictly increasing finite sequences of couples $(i, j) \in \mathbb{N}^2$ ordered anti-lexicographically:

$$(i_1, j_1) \leq (i_2, j_2) \Leftrightarrow j_1 < j_2 \text{ or } (j_1 = j_2 \text{ et } i_1 \leq i_2).$$

We suppose additionally that $\mathcal{F} \geq (0, 1) > \mathcal{G} > (0, 0)$ (thus the elements of \mathcal{G} are couples of the form $(i, 0)$, $i \in \mathbb{N}^*$, and those of \mathcal{F} are of the form (i, j) , $j \geq 1$). We say that a series $y_0 = \sum_{n \geq 1} c_n x^n \in K((x))$, $c_1 \neq 0$, is **algebraic relatively to** $(\mathcal{F}, \mathcal{G})$ if there exists a

polynomial $P(x, y) = \sum_{(i,j) \in \mathcal{F} \cup \mathcal{G}} a_{i,j} x^i y^j \in K[x, y]$ such that $P(x, y_0) = 0$.

Flajolet and Soria (see the habilitation thesis of M. Soria (1990) and [FS]) gave a closed-form expression to compute the coefficients of a formal solution of a reduced Henselian equation in the following sense:

Definition 2.2. We call **reduced Henselian equation** any equation of the following type:

$$y = Q(x, y) \text{ with } Q(x, y) \in K[x, y],$$

such that $Q(0, 0) = \frac{\partial Q}{\partial y}(0, 0) = 0$ and $Q(x, 0) \neq 0$.

Theorem 2.3 (Flajolet-Soria formula). *Let $y = Q(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ be a reduced Henselian equation. Then the coefficients of the unique solution $\sum_{n \geq 1} c_n x^n$ are given by:*

$$c_n = \sum_{m=1}^{2n-1} \frac{1}{m} \sum_{|\underline{k}|=m, \|\underline{k}\|_1=n, \|\underline{k}\|_2=m-1} \frac{m!}{\prod_{i,j} k_{i,j}!} \prod_{i,j} a_{i,j}^{k_{i,j}},$$

where $\underline{k} = (k_{i,j})_{i,j}$, $|\underline{k}| = \sum_{i,j} k_{i,j}$, $\|\underline{k}\|_1 = \sum_{i,j} i k_{i,j}$ and $\|\underline{k}\|_2 = \sum_{i,j} j k_{i,j}$.

Remark 2.4. Let us consider the particular case where the coefficients of Q verify $a_{0,j} = 0$ for all j . So, for any \underline{k} such that $|\underline{k}| = m$ and $\prod_{i,j} a_{i,j}^{k_{i,j}} \neq 0$, we have that $\|\underline{k}\|_1 \geq m$. Thus, to have $\|\underline{k}\|_1 = n$, one needs to have $m \leq n$. The Flajolet-Soria formula can be written:

$$c_n = \sum_{m=1}^n \frac{1}{m} \sum_{|\underline{k}|=m, \|\underline{k}\|_1=n, \|\underline{k}\|_2=m-1} \frac{m!}{\prod_{i,j} k_{i,j}!} \prod_{i,j} a_{i,j}^{k_{i,j}}.$$

3. CHARACTERIZING THE ALGEBRAICITY OF A FORMAL POWER SERIES

Here we resume the results from [Wil19]. Suppose we are given a series $y_0 = \sum_{n \geq 1} c_n x^n \in K((x))$ with $c_1 \neq 0$. For any $j \in \mathbb{N}$, consider the multinomial expansion of y_0^j , that we denote:

$$y_0^j = \sum_{n \geq 1} c_n^{(j)} x^n.$$

Of course, one has that $c_n^{(j)} = 0$ for $n < j$ and $c_j^{(j)} = c_1^j \neq 0$. For $j = 0$, let $y_0^0 = 1$. We remark that for any n and any j , $c_n^{(j)}$ is a homogeneous polynomial with natural number coefficients of degree j in the c_m for $m \leq n - j + 1$.

Definition 3.1. (1) Given a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we call **Wilczynski vector** $V_{i,j}$ the infinite vector with components:

- if $j \geq 1$, a sequence of i zeros followed by the coefficients of the multinomial expansion y_0^j :

$$V_{i,j} := (0, \dots, 0, c_1^{(j)}, c_2^{(j)}, \dots, c_n^{(j)}, \dots);$$

- otherwise, 1 in the i th position and 0 for the other coefficients

$$V_{i,0} := (0, \dots, 1, 0, 0, \dots, 0, \dots).$$

- (2) Let \mathcal{F} and \mathcal{G} be two sequences as in the Definition 2.1. We associate to \mathcal{F} and \mathcal{G} the **(infinite) Wilczynski matrix** whose columns are the corresponding vectors $V_{i,j}$:

$$M_{\mathcal{F},\mathcal{G}} := (V_{i,j})_{(i,j) \in \mathcal{F} \cup \mathcal{G}}.$$

We define also the **reduced Wilczynski matrix**, $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$: it is the matrix obtained from $M_{\mathcal{F},\mathcal{G}}$ by removing the columns indexed in \mathcal{G} , and also removing the corresponding rows (suppress the i th row for any $(i, 0) \in \mathcal{G}$). This amounts exactly to remove the rows containing the coefficient 1 for some Wilczynski vector indexed in \mathcal{G} .

Lemma 3.2 (Wilczynski). *The series y_0 is algebraic relatively to $(\mathcal{F}, \mathcal{G})$ if and only if all the minors of order $|\mathcal{F} \cup \mathcal{G}|$ of the Wilczynski matrix $M_{\mathcal{F},\mathcal{G}}$ vanish, or also if and only if all the minors of order $|\mathcal{F}|$ of the reduced Wilczynski matrix $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$ vanish.*

Proof. Given a non trivial polynomial $P(x, y) = \sum_{(i,j) \in \mathcal{F} \cup \mathcal{G}} a_{i,j} x^i y^j$, we compute:

$$\begin{aligned} P(x, y_0) &= \sum_{(i,j) \in \mathcal{F}} a_{i,j} x^i \left(\sum_{n \geq 1} c_n^{(j)} x^n \right) + \sum_{(i,0) \in \mathcal{G}} a_{i,0} x^i \\ &= M_{\mathcal{F},\mathcal{G}} \cdot (a_{i,j})_{(i,j) \in \mathcal{F} \cup \mathcal{G}}. \end{aligned}$$

where the components of the infinite vector thus obtained are the coefficients of the expansion of $P(x, y_0)$ with respect to the powers of x in increasing order. The series y_0 is a root of P if and only if this infinite vector is the zero vector, which means that the rank of $M_{\mathcal{F},\mathcal{G}}$ is less than $|\mathcal{F} \cup \mathcal{G}|$, the number of columns of $M_{\mathcal{F},\mathcal{G}}$. The latter condition is characterized as in finite dimension by the vanishing of all the minors of maximal order.

Let us now remark that, in the infinite vector $M_{\mathcal{F},\mathcal{G}} \cdot (a_{i,j})_{(i,j) \in \mathcal{F} \cup \mathcal{G}}$, if we remove the components of number i for $(i, 0) \in \mathcal{G}$, then we get exactly the infinite vector $M_{\mathcal{F},\mathcal{G}}^{\text{red}} \cdot (a_{i,j})_{(i,j) \in \mathcal{F}}$. The vanishing of the latter means precisely that the rank of $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$ is less than $|\mathcal{F}|$. Conversely, if the columns of $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$ are dependent for certain \mathcal{F} and \mathcal{G} , we denote by $(a_{i,j})_{(i,j) \in \mathcal{F}}$ a corresponding sequence of coefficients of a non trivial vanishing linear combination of the column vectors. Then it suffices to note that the remaining coefficients $a_{k,0}$ for $(k, 0) \in \mathcal{G}$ are each uniquely determined as follows:

$$(1) \quad a_{k,0} = - \sum_{(i,j) \in \mathcal{F}, i < k} a_{i,j} c_{k-i}^{(j)}.$$

□

We deal with the implicitization problem for algebraic power series: for fixed bounded degrees in x and y , given the expression of an algebraic series, can we reconstruct a vanishing polynomial? if yes, how?

Definition 3.3. Let us consider a reduced Wilczynski matrix $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$ associated to two sequences \mathcal{F} and \mathcal{G} of couples (i, j) as in 2.1. We call **Wilczynski polynomial** any polynomial in the coefficients c_n of y_0 obtained as a minor of $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$. We denote such Wilczynski polynomial by $Q_{\underline{k},\underline{l}}$, where $\underline{l} := ((i_1, j_1), \dots, (i_l, j_l))$ is a subsequence of \mathcal{F} indicating the l columns of $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$, and $\underline{k} := (k_1, k_2, \dots, k_l)$ a strictly increasing sequence of natural numbers indicating the l rows of $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$ used to form the minor of $M_{\mathcal{F},\mathcal{G}}^{\text{red}}$. One has that $l \in \mathbb{N}^*$, $l \leq |\mathcal{F}|$, l being the order of that minor, that we will also call the **order** of the Wilczynski polynomial $Q_{\underline{k},\underline{l}}$. Note also that a Wilczynski polynomial $Q_{\underline{k},\underline{l}}$ is homogeneous of degree

equal to $\sum_{(i,j) \in \underline{L}, c_k^{(j)} \neq 0} j$ (indeed, the coefficients of $M_{\mathcal{F}, \mathcal{G}}^{red}$ verify: $c_k^{(j)} \equiv 0 \Leftrightarrow k < j$). By convention, we call **Wilczynski polynomial of order 0** any non zero constant polynomial.

By 3.2, the algebraicity of y_0 for certain \mathcal{F} and \mathcal{G} is equivalent to the vanishing of all the $Q_{\underline{k}, \mathcal{F}}$ of order $l = |\mathcal{F}|$, for the specific values of the given c_n , coefficients of y_0 .

Example 3.4. Let $y_0 = \sum_{n \geq 1} c_n x^n \in K((x))$ be a series with $c_1 \neq 0$. We consider the conditions for y_0 to be a root of a polynomial of type:

$$P(x, y) = a_{2,0}x^2 + a_{2,1}x^2y + (a_{0,2} + a_{2,2}x^2)y^2.$$

Thus, $\mathcal{F} = \{(2, 1), (0, 2), (2, 2)\}$ and $\mathcal{G} = \{(2, 0)\}$. The corresponding Wilczynski matrix is:

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & c_1^2 & 0 \\ 0 & c_1 & 2 \cdot c_1 \cdot c_2 & 0 \\ 0 & c_2 & c_2^2 + 2 c_1 c_3 & c_1^2 \\ 0 & c_3 & 2 c_1 c_4 + 2 c_2 c_3 & 2 \cdot c_1 \cdot c_2 \\ 0 & c_4 & 2 c_2 c_4 + c_3^2 + 2 c_1 c_5 & c_2^2 + 2 c_1 c_3 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

and the reduced matrix is:

$$M^{red} := \begin{bmatrix} 0 & 0 & 0 \\ c_1 & 2 \cdot c_1 \cdot c_2 & 0 \\ c_2 & c_2^2 + 2 c_1 c_3 & c_1^2 \\ c_3 & 2 c_1 c_4 + 2 c_2 c_3 & 2 \cdot c_1 \cdot c_2 \\ c_4 & 2 c_2 c_4 + c_3^2 + 2 c_1 c_5 & c_2^2 + 2 c_1 c_3 \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

We give the four first non trivial Wilczynski polynomials of maximal order 3, equals to minors 3x3 of M^{red} . So one has that $\underline{I} = \mathcal{F}$ as index for $Q_{\underline{k}, \underline{I}}$:

$$\begin{aligned} Q_{\underline{k}, \mathcal{F}} &:= -2 c_1^2 (c_2^3 - 2 c_3 c_1 c_2 + c_1^2 c_4) \text{ pour } \underline{k} = (2, 3, 4), \\ Q_{\underline{k}, \mathcal{F}} &:= -c_1 (c_2^4 - 3 c_1^2 c_3^2 + 2 c_1^3 c_5) \text{ pour } \underline{k} = (2, 3, 5), \\ Q_{\underline{k}, \mathcal{F}} &:= -2 c_1^2 (-c_4 c_2^2 - 2 c_1 c_4 c_3 + c_2 c_3^2 + 2 c_1 c_2 c_5) \text{ pour } \underline{k} = (2, 4, 5), \\ Q_{\underline{k}, \mathcal{F}} &:= 8 c_2 c_1^2 c_4 c_3 + c_2^4 c_3 - 2 c_2^2 c_3^2 c_1 - 4 c_1^2 c_2^2 c_5 - 3 c_1^2 c_3^3 + 2 c_3 c_1^3 c_5 - 2 c_1^3 c_4^2 \\ &\quad \text{pour } \underline{k} = (3, 4, 5). \end{aligned}$$

The series y_0 is a root of a polynomial $P(x, y)$ as above if and only if all the Wilczynski polynomials of order 3 vanish. This implies in particular that:

$$c_4 = -\frac{c_2(c_2^2 - 2 c_1 c_3)}{c_1^2} \text{ and } c_5 = -\frac{c_2^4 - 3 c_1^2 c_3^2}{2 c_1^3}.$$

Theorem 3.5. Let \mathcal{F} and \mathcal{G} be two finite sequences of couples as in 2.1. We set $d_y := \max\{j, (i, j) \in \mathcal{F}\}$, $d_x := \max\{i, (i, j) \in \mathcal{F} \cup \mathcal{G}\}$ and $N := 2d_x d_y$. Then there exists a finite number of homogeneous polynomials $a_{i,j}^{(\lambda)} \in \mathbb{Z}[x_1, \dots, x_N]$, $(i, j) \in \mathcal{F} \cup \mathcal{G}$, $\lambda \in \Lambda$, of total degree $\deg a_{i,j}^{(\lambda)} \leq \frac{1}{2}d_y(d_y+1)(d_x+1)-1$ for $(i, j) \in \mathcal{F}$, and $\deg a_{i,0}^{(\lambda)} \leq \frac{1}{2}d_y(d_y+1)(d_x+1)-1+i$

for $(i, 0) \in \mathcal{G}$, such that, for any $y_0 = \sum_{n \geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$ series algebraic relatively to $(\mathcal{F}, \mathcal{G})$, there is $\lambda \in \Lambda$ such that the polynomial:

$$P^{(\lambda)}(x, y) = \sum_{(i,j) \in \mathcal{F}} a_{i,j}^{(\lambda)}(c_1, \dots, c_N) x^i y^j + \sum_{(i,0) \in \mathcal{G}} a_{i,0}^{(\lambda)}(c_1, \dots, c_N) x^i \in K[x, y]$$

vanishes at y_0 .

First, we give the reconstruction process. Then we will show its finiteness.

Proof. Let $y_0 = \sum_{n \geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$ be algebraic relatively to $(\mathcal{F}, \mathcal{G})$. We show how to reconstruct a vanishing polynomial $P(x, y)$ of y_0 .

Let $Q(x, y) = \sum_{(i,j) \in \mathcal{F}} b_{i,j} x^i y^j + \sum_{(i,0) \in \mathcal{G}} b_{i,0} x^i$ be a polynomial that vanishes at y_0 . We proceed by induction on m the number of non zero coefficients $b_{i,j}$ for $(i, j) \in \mathcal{F}$. If $m = 1$, $Q(x, y)$ is of the form:

$$Q(x, y) = b_{i,j} x^i y^j + \sum_{(i,0) \in \mathcal{G}} b_{i,0} x^i,$$

with $b_{i,j} \neq 0$. So we must have that $b_{n,0} = 0$ for $n < i + j$, and the series y_0 verifies:

$$\sum_{(n,0) \in \mathcal{G}} b_{n,0} x^n = -b_{i,j} x^i y_0^j = \sum_{n \geq i} -b_{i,j} c_{n-i}^{(j)} x^n.$$

The criterion 3.2 means here that the order 1 minors of $M_{(i,j),\mathcal{G}}^{red}$, being equal to $c_{n-i}^{(j)}$ for $(n, 0) \notin \mathcal{G}$, are all null. We fix the coefficient $a_{i,j}$ arbitrarily in $\mathbb{Z} \setminus \{0\}$: it is a constant Wilczynski polynomial. Then the other coefficients are uniquely determined in accordance with the relation (1) by the equation:

$$a_{n,0} := -a_{i,j} c_{n-i}^{(j)}, \quad (n, 0) \in \mathcal{G}.$$

Thus the coefficient $a_{n,0}$ is a polynomial of degree j in the c_k , $k \leq n - i - j + 1$, which verifies indeed that $j \leq d_y \leq \frac{1}{2} d_y (d_y + 1) (d_x + 1) \leq \frac{1}{2} d_y (d_y + 1) (d_x + 1) - 1 + n$. Consider now the case where the vanishing polynomial $Q(x, y)$ of y_0 has $m \geq 2$ non zero terms. So there exists a subfamily $\mathcal{F}' \subset \mathcal{F}$, the indices of the non zero coefficients of $Q(x, y)$, with $|\mathcal{F}'| = m$ and such that the order m minors of $M_{\mathcal{F}',\mathcal{G}}^{red}$ all vanish. Suppose that there exists an order $m - 1$ minor of this matrix, i.e. a Wilczynski polynomial $Q_{\underline{k}_0, \underline{L}_0}$, which is non zero. Denote by $M_{\underline{k}_0, \underline{L}_0}$ the square matrix whose determinant is this minor, and $C_{\underline{k}_0, (i_0, j_0)}$ the p_0 -th column that has been removed to form this minor. We get a Cramer system of equations:

$$M_{\underline{k}_0, \underline{L}_0} \cdot (b_{i,j})_{(i,j) \neq (i_0, j_0)} = -b_{i_0, j_0} C_{\underline{k}_0, (i_0, j_0)}.$$

Let us build the coefficients $a_{i,j}$ verifying:

$$M_{\underline{k}_0, \underline{L}_0} \cdot (a_{i,j})_{(i,j) \neq (i_0, j_0)} = -a_{i_0, j_0} C_{\underline{k}_0, (i_0, j_0)},$$

by taking $a_{i_0, j_0} := (-1)^{p_0} Q_{\underline{k}_0, \underline{L}_0}$ and by computing the other $a_{i,j}$ by Cramer's rule. Thus the $a_{i,j}$ are all order $m - 1$ minors of $M_{\mathcal{F}',\mathcal{G}}^{red}$, and so, up to the sign, Wilczynski polynomials $Q_{\underline{k}_0, \underline{L}}$ of order $m - 1$. If $\underline{k}_0 = (k_{0,1}, \dots, k_{0,m-1})$, we set:

$$(2) \quad N_{y_0} := k_{0,m-1}.$$

The $a_{i,j}$ are homogeneous polynomials of $\mathbb{Z}[x_1, \dots, x_{N_{y_0}}]$. The degree of a Wilczynski polynomial $Q_{\underline{k}_0, \underline{l}}$ verifies:

$$\begin{aligned} \deg Q_{\underline{k}_0, \underline{l}} &= \sum_{(i,j) \in \underline{l}, c_k^{(j)} \neq 0} j \\ &\leq -1 + \sum_{(i,j) \in \mathcal{F}} j \\ &\leq -1 + (d_x + 1) \sum_{j=1}^{d_y} j \\ &= \frac{1}{2} d_y (d_y + 1) (d_x + 1) - 1. \end{aligned}$$

The coefficients $a_{n,0}$ for $(n, 0) \in \mathcal{G}$ are obtained via the relations (1):

$$a_{n,0} = - \sum_{(i,j) \in \mathcal{F}, n > i} a_{i,j} c_{n-i}^{(j)}.$$

Knowing that $c_{n-i}^{(j)} \neq 0 \Rightarrow n - i \geq j$, and in this case $\deg c_{n-i}^{(j)} = j$, we deduce that $\deg a_{n,0} \leq n + \max_{(i,j) \in \mathcal{F}} (\deg a_{i,j})$ as desired. The polynomial $P(x, y) = \sum_{(i,j) \in \mathcal{F}' \cup \mathcal{G}} a_{i,j} x^i y^j$

is proportional to Q , so it vanishes at y_0 .

Suppose now that all the minors of order $m - 1$ are zero. So, restricting to a subfamily $\mathcal{F}'' \subset \mathcal{F}'$ of $m - 1$ vectors among the m Wilczynski vectors $V_{i,j}$, $(i, j) \in \mathcal{F}'$, with the same family \mathcal{G} , one has a new reduced Wilczynski matrix, with $m - 1$ columns, all of which minors of order $m - 1$ are null. So y_0 is algebraic relatively to $(\mathcal{F}'', \mathcal{G})$. By induction on m , one has reconstructed a polynomial $P(x, y)$ vanishing at y_0 .

To obtain the Theorem 3.5, it suffices now to show that there exists a uniform bound N_{d_x, d_y} for the depth in $M_{\mathcal{F}, \mathcal{G}}^{\text{red}}$ to which we get the reconstruction process, that is, the depth at which we find a first non zero minor. We reach this in the two following lemmas.

Lemma 3.6. *Let $d_x, d_y \in \mathbb{N}^*$. For any series $y_0 = \sum_{n \geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$, verifying an equation $P(x, y_0) = 0$ where $P(x, y) \in K[x, y]$, $\deg_x P \leq d_x$, $\deg_y P \leq d_y$, and for any polynomial $Q(x, y) \in K[x, y]$, $\deg_x Q \leq d_x$, $\deg_y Q \leq d_y$, such that $Q(x, y_0) \neq 0$, one has that $\text{ord}_x Q(x, y_0) \leq 2d_x d_y$.*

Proof. Let y_0 be a series as in the statement of Lemma 3.6. We consider the ideal $I_0 := \{R(x, y) \in K[x, y] \mid R(x, y_0) = 0\}$. By assumption, it is a non trivial prime ideal, so its height is one or two. If it were equal to 2, then it would be a maximal ideal. But I_0 is included into the ideal $\{R(x, y) \in K[x, y] \mid R(0, 0) = 0\}$, so:

$$I_0 = \{R(x, y) \in K[x, y] \mid R(0, 0) = 0\} = (x, y)$$

which is absurd because $x \notin I_0$. So, I_0 is a height one prime ideal of the factorial ring $K[x, y]$. It is generated by an irreducible polynomial $P_0(x, y) \in K[x, y]$. We set $d_{0,x} := \deg_x P_0$ and $d_{0,y} := \deg_y P_0$. Note also that, by factoriality of $K[x, y]$, P_0 is also irreducible as an element of $K(x)[y]$.

Let P be as in the statement of Lemma 3.6. One has that $P = S P_0$ for some $S \in K[x, y]$. Hence $d_{0,x} \leq d_x$ and $d_{0,y} \leq d_y$. Let $Q \in K[x, y]$ such that $Q(x, y_0) \neq 0$ with $\deg_x Q \leq d_x$, $\deg_y Q \leq d_y$. So P_0 and Q are coprime in $K(x)[y]$. Their resultant $r(x)$ is non zero. One has the following Bézout relation in $K[x][y]$:

$$A(x, y)P_0(x, y) + B(x, y)Q(x, y) = r(x).$$

We evaluate at $y = y_0$:

$$0 + B(x, y_0)Q(x, y_0) = r(x).$$

So $\text{ord}_x Q(x, y_0) \leq \deg_x r(x)$. But, the resultant is a determinant of order at most $d_y + d_{0,y} \leq 2d_y$ whose entries are polynomials in $K[x]$ of degree at most $\max\{d_x, d_{0,x}\} \leq d_x$. So, $\deg_x r(x) \leq 2d_x d_y$. Hence, one has that: $\text{ord}_x Q(x, y_0) \leq 2d_x d_y$. \square

Lemma 3.7. *Let $\mathcal{F}' \subseteq \mathcal{F}$. If y_0 is not algebraic relatively to $(\mathcal{F}', \mathcal{G})$, we denote $l := |\mathcal{F}'|$ and $p := \min\{k_l \mid Q_{\underline{k}, \mathcal{F}'} \neq 0, \underline{k} = (k_1, \dots, k_l)\}$. Then, for any polynomial $Q(x, y) =$*

$$\sum_{(i,j) \in \mathcal{F}' \cup \mathcal{G}} b_{i,j} x^i y^j, \text{ we have:}$$

$$\text{ord}_x Q(x, y_0) \leq p \leq 2d_x d_y,$$

and the value p is reached for a certain polynomial Q_0 .

Proof. By the definition of p , for any $\underline{k} = (k_1, \dots, k_l)$ with $k_l < p$, we have that $Q_{\underline{k}, \mathcal{F}'} = 0$. This means that the rank of the column vectors $V_{i,j,p-1}$ that are the restrictions of those of $M_{\mathcal{F}', \mathcal{G}}^{\text{red}}$ up to the line $p-1$, is less than $l = |\mathcal{F}'|$. There are coefficients $(a_{i,j})_{(i,j) \in \mathcal{F}' \cup \mathcal{G}}$ not all zero such that $\sum_{(i,j) \in \mathcal{F}' \cup \mathcal{G}} a_{i,j} V_{i,j,p-1} = (0)$, which is equivalent to the vanishing of the $p-1$

first terms of $Q_0(x, y_0) := \sum_{(i,j) \in \mathcal{F}' \cup \mathcal{G}} a_{i,j} x^i y_0^j$. Thus, $\text{ord}_x Q_0(x, y_0) \geq p$, and so $p \leq 2d_x d_y$.

On the other hand, again by the definition of p , the column vectors up to the line p are, in turn, of rank $l = |\mathcal{F}'|$. Any non trivial linear combination is non null, so $\text{ord}_x Q(x, y_0) \leq p$ for all $Q(x, y) := \sum_{(i,j) \in \mathcal{F}' \cup \mathcal{G}} b_{i,j} x^i y^j$. \square

We achieve the proof of Theorem 3.5 via the Lemmas 3.6 and 3.7 by considering for a given algebraic series y_0 a family $\mathcal{F}'' \subset \mathcal{F}$ minimal among the families such that y_0 is algebraic relatively to $(\mathcal{F}'', \mathcal{G})$. Hence, the natural number N_{y_0} of (2) is always bounded by $N = 2d_x d_y$. \square

Construction of the coefficients $a_{i,j}^{(A_0)}$ for a given y_0 .

Let y_0 be algebraic relatively to $(\mathcal{F}, \mathcal{G})$ as in 2.1. Let $N = 2d_x d_y$ as in 3.5. We denote by M_N the matrix consisting in the N first lines of $M_{\mathcal{F}, \mathcal{G}}^{\text{red}}$. Let r be the rank of M , and $m := r + 1$. The minors of M of order m are all zero and there exists a minor of order $m - 1 = r$ which is non zero. There are two cases. If $r = 0$, we choose $(i, j) \in \mathcal{F}$ and we fix the coefficients $a_{i,j} := 1$ and $a_{l,m} = 0$ pour $(l, m) \in \mathcal{F}$, $(l, m) \neq (i, j)$. Then we derive the coefficients $a_{i,0}$ for $(i, 0) \in \mathcal{G}$ from the relations (1). The polynomials P thus obtained are all annihilators of y_0 .

If $r \geq 1$, we consider all the Wilczynski polynomials $Q_{\underline{k}, \underline{l}}$ of order r that do not vanish when evaluated at c_1, \dots, c_N . Each of them allows to reconstruct coefficients $a_{i,j}^{(A)}$, $(i, j) \in \mathcal{F}$, and subsequently coefficients $a_{i,0}$, $(i, 0) \in \mathcal{G}$, via (1). The corresponding polynomials $P^{(A)}$ are annihilators of y_0 if and only if $\text{ord}_x P^{(A)} \left(x, \sum_{k=1}^N c_k x^k \right) > N$.

Example 3.8. We resume the Example 3.4, and note that, for $\underline{k} = (2, 3)$ and for $\underline{l} = ((2, 1), (2, 2))$, we have that:

$$Q_{\underline{k}, \underline{l}} = \begin{vmatrix} c_1 & 0 \\ c_2 & c_1^2 \end{vmatrix} = c_1^3 \neq 0.$$

So we set $a_{0,2} := (-1)^2 c_1^3 = c_1^3$ and, applying the Cramer's rule :

$$\begin{cases} a_{2,1} &:= (-1)^1 \begin{vmatrix} 2 \cdot c_1 \cdot c_2 & 0 \\ c_2^2 + 2 c_1 c_3 & c_1^2 \end{vmatrix} = -2c_1^3 c_2 \\ a_{2,2} &:= (-1)^3 \begin{vmatrix} c_1 & 2 \cdot c_1 \cdot c_2 \\ c_2 & c_2^2 + 2 c_1 c_3 \end{vmatrix} = c_1 (c_2^2 - 2c_1 c_3). \end{cases}$$

We deduce from the formulas (1) that:

$$a_{2,0} = -a_{2,1} \cdot 0 - a_{0,2} \cdot c_1^2 - a_{2,2} \cdot 0 = -c_1^5.$$

A vanishing polynomial of a series $y_0 = \sum_{n \geq 1} c_n x^n \in K((x))$, $c_1 \neq 0$, algebraic relatively to $\mathcal{F} = ((2, 1), (0, 2), (2, 2))$ and $\mathcal{G} = (2, 0)$ is:

$$\begin{aligned} P(x, y) &= -c_1^5 x^2 - 2c_1^3 c_2 x^2 y + c_1^3 y^2 + c_1 (c_2^2 - 2c_1 c_3) x^2 y^2 \\ &= c_1 \left[-c_1^4 x^2 - 2c_1^2 c_2 x^2 y + c_1^2 y^2 + (c_2^2 - 2c_1 c_3) x^2 y^2 \right]. \end{aligned}$$

Remark 3.9. (1) Let y_0 be a series algebraic with vanishing polynomial of degree d_x in x and d_y in y . According to [BCG⁺14, Chap. 7], the method of reconstruction of equation based on Padé-Hermite approximants provides a priori only polynomials $P(x, y) = \sum_{i \leq d_x, j \leq d_y} a_{i,j} x^i y^j$ such that $P(x, y_0) \equiv 0 [x^\sigma]$ with $\sigma = (d_x + 1)(d_y + 1) - 1$.

Subsequently, one has to check whether $P(x, y_0) = 0$ actually. By our Lemma 3.6, one can always certify that $P(x, y_0) = 0$ just by verifying that $P(x, y_0) \equiv 0 [x^\tau]$ with $\tau = 2d_x d_y$. Hence this reconstruction method as implemented in the GFUN package in Maple software holds for any equation of degree less than d_x in x and d_y in y , not for only irreducible ones as in [BCG⁺14, Theorem 8, p. 110].

(2) Let us consider the case where y_0 is a rational fraction:

$$\begin{aligned} y_0 &= \frac{-a_0(x)}{a_1(x)} = \frac{-a_{1,0}x - \dots - a_{d_0,0}x^{d_0}}{1 + a_{1,1}x + \dots + a_{d_1,1}x^{d_1}} \\ &= \sum_{n \geq 1} c_n x^n \text{ with } c_1 \neq 0. \end{aligned}$$

Thus, y_0 is algebraic relatively to $\mathcal{F} = \{(0, 1), \dots, (d_1, 1)\}$ and $\mathcal{G} = \{(1, 0), \dots, (d_0, 0)\}$. The Wilczynski polynomials of order $|\mathcal{F}| = d_1 + 1$ are all null. The Wilczynski polynomial $Q_{\underline{k}_0, \underline{l}_0}$ of order d_1 with $\underline{k}_0 = (1, \dots, d_1)$ and $\underline{l}_0 = ((1, 1), \dots, (d_1, 1))$ is equal, up to the sign, to the resultant of $a_0(x)$ and $a_1(x)$, by [GKZ94, chap 12 (1.15) p 401].

(3) In the present section, the field K can be of any characteristic.

4. CLOSED-FORM EXPRESSION OF AN ALGEBRAIC SERIES.

Let us assume from now on that K has zero characteristic. Our purpose is to determine the coefficients of an algebraic series in terms of the coefficients of a vanishing polynomial.

We consider the following polynomial of degrees bounded by d_x in x and d_y in y :

$$\begin{aligned} P(x, y) &= \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} a_{i,j} x^i y^j, \text{ with } P(x, y) \in K[x, y] \\ &= \sum_{i=0}^{d_x} \pi_i(y) x^i \\ &= \sum_{j=0}^{d_y} a_j(x) y^j, \end{aligned}$$

and a formal power series which is a simple root:

$$y_0 = \sum_{n \geq 1} c_n x^n, \text{ with } y_0 \in K[[x]], \ c_1 \neq 0.$$

The field $K((x))$ is endowed with the x -adic valuation ord_x .

Classically (e.g. [Wal78]), the resolution of $P = 0$ with the Newton-Puiseux method is algorithmic, with two stages:

- (1) a first stage of separation of the branches solutions, which illustrates the following fact: y_0 may share a principal part with other roots of P . This is equivalent to the fact that this principal part is also the principal part of a root of $\partial P / \partial y$.
- (2) a second stage of unique "automatic" resolution: once the branches are separated, the remaining part of y_0 is a root of an equation called Henselian in the formal valued context (y_0 seen as an algebraic formal power series), and called of implicit function type in the context of differentiable functions (y_0 seen as the convergent Taylor expansion of an algebraic function).

We give here a version of the algebraic content of this algorithmic resolution.

Notation 4.1. For any $k \in \mathbb{N}$ and for $Q(x, y) = \sum_{j=0}^d a_j(x) y^j \in K((x))[y]$, we denote:

- $\text{ord}_x Q := \min\{\text{ord}_x a_j(x), \ j = 0, \dots, d\}$
- $z_0 := 0$ and for $k \geq 1$, $z_k := \sum_{n=1}^k c_n x^n$
- $y_k := y_0 - z_k = \sum_{n \geq k+1} c_n x^n$
- $Q_k(x, y) := Q(z_k + x^{k+1}y) = \sum_{i=i_k}^{d_k} \pi_{k,i}(y) x^i$ where $i_k = \text{ord}_x Q_k$ and $d_k := \deg_x Q_k$

Lemma 4.2. (1) The series y_0 is a root of $P(x, y)$ if and only if the sequence $(i_k)_{k \in \mathbb{N}^*}$ is strictly increasing where $i_k = \text{ord}_x P_k$.
(2) The series y_0 is a simple root of $P(x, y)$ if and only if the sequence $(i_k)_{k \in \mathbb{N}^*}$ is strictly increasing and there exists a lowest index k_0 such that $i_{k_0+1} = i_{k_0} + 1$. In that case, one has that $i_{k+1} = i_k + 1$ for any $k \geq k_0$.

Proof. (1) Note that for any k , $i_k \leq \text{ord}_x P_k(x, 0) = \text{ord}_x P(x, z_k)$. Hence, if the sequence $(i_k)_{k \in \mathbb{N}^*}$ is strictly increasing, it tends to $+\infty$, and so does $\text{ord}_x P(x, z_k)$. y_0 is indeed a root of $P(x, y)$. Reciprocally, suppose that there exists $1 \leq k < l$ such that $i_k \geq i_l$. We apply the Taylor formula to $P_j(x, y)$ for $j > k$:

$$\begin{aligned}
(3) \quad P_j(x, y) &= P_k(x, c_{k+1} + c_{k+2}x + \cdots + x^{j-k}y) \\
&= \pi_{k,i_k}(c_{k+1})x^{i_k} + \left[\pi'_{k,i_k}(c_{k+1})c_{k+2} + \pi_{k,i_k+1}(c_{k+1}) \right] x^{i_k+1} + \cdots
\end{aligned}$$

For $1 \leq k \leq j \leq l$, $i_l \geq i_j \geq i_k$, so $i_j = i_k$. Thus, $\pi_{k,i_k}(c_{k+1}) \neq 0$, so for any $j > k$, $\text{ord}_x P_j(x, 0) = \text{ord}_x P(x, z_j) = i_k$. Hence $\text{ord}_x P(x, y_0) = i_k \neq +\infty$.

(2) The series y_0 is a double root of P if and only if it is a root of P and $\partial P / \partial y$. We apply the Taylor formula for certain $k \in \mathbb{N}^*$:

$$\begin{aligned}
(4) \quad P_{k+1}(x, y) &= P_k(x, c_{k+1} + xy) \\
&= \pi_{k,i_k}(c_{k+1})x^{i_k} + \left[\pi'_{k,i_k}(c_{k+1})y + \pi_{k,i_k+1}(c_{k+1}) \right] x^{i_k+1} \\
&\quad + \left[\frac{\pi''_{k,i_k}(c_{k+1})}{2}y^2 + \pi'_{k,i_k+1}(c_{k+1})y + \pi_{k,i_k+2}(c_{k+1}) \right] x^{i_k+2} + \cdots
\end{aligned}$$

Note that:

$$\frac{\partial P_k}{\partial y}(x, y) = x^{k+1} \left(\frac{\partial P}{\partial y} \right)_k(x, y) = \sum_{i=i_k}^{d_k} \pi'_{k,i}(y) x^i$$

One has that $\pi_{k,i_k} \neq 0$ and $\pi_{k,i_k}(c_{k+1}) = 0$ (see the point (1) above), so $\pi'_{k,i_k}(y) \neq 0$. Thus

$$\begin{aligned}
\text{ord}_x \left(\frac{\partial P}{\partial y} \right)_k &= i_k - k - 1. \text{ We perform the Taylor expansion of } \left(\frac{\partial P}{\partial y} \right)_{k+1} = \left(\frac{\partial P}{\partial y} \right)_k(c_{k+1} + xy): \\
\left(\frac{\partial P}{\partial y} \right)_{k+1}(x, y) &= \pi'_{k,i_k}(c_{k+1})x^{i_k-k-1} + \left[\pi''_{k,i_k}(c_{k+1})y + \pi'_{k,i_k+1}(c_{k+1}) \right] x^{i_k-k} + \cdots
\end{aligned}$$

By the point (1), if y_0 is a double root P , we must have $\pi'_{k,i_k}(c_{k+1}) = 0$. Moreover, if $\pi_{k,i_k+1}(c_{k+1}) \neq 0$, we would have $i_{k+1} = i_k + 1$ and even $i_{k+j} = i_k + 1$ for every j according to (3): y_0 could not be a root of P . So, $\pi_{k,i_k+1}(c_{k+1}) = 0$, and, accordingly, $i_{k+1} \geq i_k + 2$.

If y_0 is a simple root of P , from the point (1) there exists a lowest natural number k_0 such that the sequence $(i_k - k - 1)_{k \in \mathbb{N}^*}$ is no longer strictly increasing, that is, such that

$$\begin{aligned}
\pi'_{k_0,i_{k_0}}(c_{k_0+1}) &\neq 0. \text{ For any } k \geq k_0, \text{ we consider the Taylor expansion of } \left(\frac{\partial P}{\partial y} \right)_{k+1} = \\
\left(\frac{\partial P}{\partial y} \right)_{k_0}(c_{k_0+1} + \cdots + x^{k-k_0+1}y):
\end{aligned}$$

$$(5) \quad \left(\frac{\partial P}{\partial y} \right)_{k+1}(x, y) = \pi'_{k_0,i_{k_0}}(c_{k_0+1})x^{i_{k_0}-k_0-1} + \left[\pi''_{k_0,i_{k_0}}(c_{k_0+1})c_{k_0+2} + \pi'_{k_0,i_{k_0}+1}(c_{k_0+1}) \right] x^{i_{k_0}-k_0} + \cdots$$

and we get that:

$$(6) \quad \text{ord}_x \left(\frac{\partial P}{\partial y} \right)_{k+1}(x, 0) = \text{ord}_x \left(\frac{\partial P}{\partial y} \right)_{k+1} = i_{k_0} - k_0 - 1.$$

As $\pi'_{k+1,i_{k+1}}(y) \neq 0$, we obtain that $i_{k+1} = \text{ord}_x P_{k+1} = \text{ord}_x \left(\frac{\partial P_{k+1}}{\partial y} \right) = k + 2 + \text{ord}_x \left(\frac{\partial P}{\partial y} \right)_{k+1} = i_{k_0} + k - k_0 + 1$. Hence, from the rank k_0 , the sequence (i_k) increases one by one. \square

Resuming the notations of the Theorem 3.5 and of the Lemma 4.2, the natural number k_0 represents the length of the principal part in the stage of separation of the branches. In the following lemma, we bound it using the Lemma 3.6 or the discriminant Δ_P of P .

Lemma 4.3. *With the notations of the Theorem 3.5, the natural number k_0 verifies that:*

$$k_0 \leq 2d_x d_y + 1.$$

In particular, if P has only simple roots:

$$k_0 \leq d_x(2d_y - 1) + 1.$$

Proof . By the Lemma 3.6, since $P(x, y_0) = 0$ and $\frac{\partial P}{\partial y}(x, y_0) \neq 0$, one has that:

$$\text{ord}_x \frac{\partial P}{\partial y}(x, y_0) \leq 2d_x d_y.$$

But, by definition, k_0 is the lowest natural number such that:

$$\text{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0+1}) = \text{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0}) = i_{k_0} - k_0 - 1$$

(see the point (2) of the preceding lemma). Hence, we get from (6) that $\text{ord}_x \frac{\partial P}{\partial y}(x, z_k) = \text{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0}) = i_{k_0} - k_0 - 1$ for any $k > k_0$. So, $\text{ord}_x \frac{\partial P}{\partial y}(x, y_0) = \text{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0})$. To conclude, we note that $\text{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0}) \geq k_0 - 1$ by definition of k_0 .

In the case where P has only simple roots, as in the proof of the Lemma 3.6, $\text{ord}_x \frac{\partial P}{\partial y}(x, y_0)$ is bounded by the degree of the resultant of P and $\frac{\partial P}{\partial y}$, say the discriminant Δ_P of P , which is bounded by $d_x(2d_y - 1) + 1$. \square

Theorem 4.4. Consider the following polynomial in $K[x, y]$ of given degrees d_x in x and d_y in y :

$$P(x, y) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} a_{i,j} x^i y^j = \sum_{i=0}^{d_x} \pi_i(y) x^i,$$

and a formal power series which is a simple root:

$$y_0 = \sum_{n \geq 1} c_n x^n \in K[[x]], \quad c_1 \neq 0.$$

Resuming the notations of 4.1 and 4.2, we set $\omega_0 := \pi'_{k_0, i_{k_0}}(c_{k_0+1}) \neq 0$. Hence, for any $k > k_0$:

- either the polynomial $z_{k+1} = \sum_{n=1}^{k+1} c_n x^n$ is a solution of $P(x, y) = 0$;
- or the polynomial $R_k(x, y) := \frac{P_k(x, y + c_{k+1})}{-\omega_0 x^{i_k}} = -y + Q_k(x, y)$ defines a reduced Henselian equation:

$$y = Q_k(x, y)$$

with $Q_k(0, y) \equiv 0$ and satisfied by:

$$t_{k+1} := \frac{y_0 - z_{k+1}}{x^{k+1}} = c_{k+2}x + c_{k+3}x^2 + \dots$$

Proof . We show by induction on $k > k_0$ that $R_k(x, y) = -y + xT_k(x, y)$ with $T_k(x, y) \in K[x, y]$. For $k = k_0 + 1$, by (4), since $i_{k_0+1} = i_{k_0} + 1$, we have that:

$$P_{k_0+1}(x, y) = [\omega_0 y + \pi_{k_0, i_{k_0}+1}(c_{k_0+1})] x^{i_{k_0}+1} + \dots$$

Since $i_{k_0+2} = i_{k_0} + 2$, $\pi_{k_0+1, i_{k_0}+1}(y) = \omega_0 y + \pi_{k_0, i_{k_0}+1}(c_{k_0+1})$ vanishes at c_{k_0+2} , which implies that $c_{k_0+2} = \frac{-\pi_{k_0, i_{k_0}+1}(c_{k_0+1})}{\omega_0}$. Computing $R_{k_0+1}(x, y)$, it follows that:

$$R_{k_0+1}(x, y) = -y + Q_{k_0+1}(x, y) \text{ with}$$

$$Q_{k_0+1}(x, y) = x \left[\frac{\pi''_{k, i_{k_0}}(c_{k_0+1})}{2} (y + c_{k_0+2})^2 + \pi'_{k_0, i_{k_0}+1}(c_{k_0+1}) (y + c_{k_0+2}) + \pi_{k_0, i_{k_0}+2}(c_{k_0+1}) \right] + x^2[\dots].$$

So $Q_{k_0+1}(0, y) \equiv 0$.

Suppose that the property holds true at a rank $k > k_0 + 1$. It follows that:

$$\begin{aligned} P_k(x, y) &= \omega_0(y - c_{k+1})x^{i_k} + x^{i_k+1}\tilde{T}_k(x, y) \\ &= \pi_{k, i_k}(y)x^{i_k} + \pi_{k, i_k+1}(y)x^{i_k+1} + \dots \end{aligned}$$

Since $P_{k+1}(x, y) = P_k(x, c_{k+1} + xy)$, we have that:

$$P_{k+1}(x, y) = [\omega_0 y + \pi_{k, i_k+1}(c_{k+1})]x^{i_k+1} + \pi_{k+1, i_k+2}(y)x^{i_k+2} + \dots$$

But $i_k + 2 = i_{k+2} > i_{k+1} = i_k + 1$. So we must have $\pi_{k+1, i_k+1}(c_{k+2}) = 0$. So, $c_{k+2} = \frac{-\pi_{k, i_k+1}(c_{k+1})}{\omega_0}$. It follows that:

$$P_{k+1}(x, y) = \omega_0(y - c_{k+2})x^{i_k+1} + \pi_{k+1, i_k+2}(y)x^{i_k+2} + \dots,$$

Hence:

$$\begin{aligned} R_{k+1}(x, y) &= -y - x \frac{\pi_{k+1, i_k+2}(y + c_{k+2})}{\omega_0} + x^2[\dots] + \dots \\ &= -y + xT_{k+1}(x, y), \quad T_{k+1} \in K[x, y], \end{aligned}$$

as desired.

In particular, $Q_k(0, 0) = \frac{\partial Q_k}{\partial y}(0, 0) = 0$. So the equation $y = Q_k(x, y)$ is reduced Henselian if and only if $Q_k(x, 0) \neq 0$, which is equivalent to z_{k+1} not being a root of P . \square

Remark 4.5. By (5), we note that

$$\left(\frac{\partial P}{\partial y} \right)(x, y_0) = \omega_0 x^{i_{k_0} - k_0 - 1} + \dots$$

Thus, ω_0 is the initial coefficient of $\left(\frac{\partial P}{\partial y} \right)(x, y_0)$.

For the courageous reader, in the case where y_0 is a series which is not a polynomial, we deduce from 4.4 and from the Flajolet-Soria formula 2.3 a closed-form expression for the coefficients of y_0 in terms of the coefficients $a_{i,j}$ of P and of the coefficients of an initial part z_k of y_0 sufficiently large.

Corollary 4.6. *For any $k \geq k_0 + 1$, for any $p \geq 1$, one has that:*

$$c_{k+1+p} = \sum_{q=1}^p \frac{1}{q} \left(\frac{-1}{\omega_0} \right)^q \sum_{|S|=q, \|S\|_2 \geq q-1} A^S \left(\sum_{\substack{|T_S|=|S| \|S\|_2 - q + 1 \\ \|T_S\| = p + q i_k - (q-1)(k+1) - \|S\|_1}} e_{T_S} C^{T_S} \right),$$

where $S = (s_{i,j})$, $A^S = \prod_{i=0,\dots,d_x, j=0,\dots,d_y} a_{i,j}^{s_{i,j}}$, $T_S = (t_{S,i})$, $C^{T_S} = \prod_{i=1}^{k+1} c_i^{t_{S,i}}$, and $e_{T_S} \in \mathbb{N}$ is of the form:

$$e_{T_S} = \sum_{\substack{n_{i,j,L}^{l,m} \\ l=1,\dots,(k+1)d_y+d_x-i_k \\ m=0,\dots,m_l}} \frac{q!}{\prod_{i=0,\dots,d_x, j=m,\dots,d_y} n_{i,j,L}^{l,m}!} \prod_{\substack{l=1,\dots,(k+1)d_y+d_x-i_k \\ m=0,\dots,m_l}} \prod_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \prod_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \left(\frac{j!}{m! L!} \right)^{n_{i,j,L}^{l,m}},$$

where $m_l := \min \left\{ \left\lfloor \frac{l+i_k}{k+1} \right\rfloor, d_y \right\}$, $L = L_{i,j}^{l,m} = (l_{i,j,1}^{l,m}, \dots, l_{i,j,k+1}^{l,m})$, and where the sum is taken over the set of $(n_{i,j,L}^{l,m})_{\substack{l=1,\dots,(k+1)d_y+d_x-i_k, m=0,\dots,m_l \\ i=0,\dots,d_x, j=m,\dots,d_y, |L|=j-m, \|L\|=l+i_k-m(k+1)-i}}$ such that:

$$\sum_{l,m} \sum_{i,j} \sum_L n_{i,j,L}^{l,m} = q \quad \text{and} \quad \sum_{l,m} \sum_{i,j} \sum_L n_{i,j,L}^{l,m} L = T_S.$$

Proof. We get started by computing the coefficients of $\omega_0 x^{i_k} R_k$, in order to get those of Q_k :

$$\begin{aligned} \omega_0 x^{i_k} R_k &= P_k(x, y + c_{k+1}) \\ &= P(x, z_{k+1} + x^{k+1}y) \\ &= \sum_{i=0,\dots,d_x, j=0,\dots,d_y} a_{i,j} x^i (z_{k+1} + x^{k+1}y)^j \\ &= \sum_{i=0,\dots,d_x, j=0,\dots,d_y} a_{i,j} x^i \sum_{m=0}^j \frac{j!}{m! (j-m)!} z_{k+1}^{j-m} x^{m(k+1)} y^m. \end{aligned}$$

For $L = (l_1, \dots, l_{k+1})$, we denote $C^L := c_1^{l_1} \dots c_{k+1}^{l_{k+1}}$. One has that:

$$z_{k+1}^{j-m} = \sum_{|L|=j-m} \frac{(j-m)!}{L!} C^L x^{\|L\|}.$$

So:

$$\omega_0 x^{i_k} R_k = \sum_{m=0}^{d_y} \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} a_{i,j} \sum_{|L|=j-m} \frac{j!}{m! L!} C^L x^{\|L\|+m(k+1)+i} y^m.$$

We set $\hat{l} = \|L\| + m(k+1) + i$, which ranges between $m(k+1)$ and $(k+1)(d_y-m) + m(k+1) + d_x = (k+1)d_y + d_x$. Thus:

$$\omega_0 x^{i_k} R_k = \sum_{\substack{m=0,\dots,d_y \\ \hat{l}=m(k+1), \dots, (k+1)d_y+d_x}} \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} a_{i,j} \sum_{\substack{|L|=j-m \\ \|L\|=l-i \\ l=\hat{l}-m(k+1)-i}} \frac{j!}{m! L!} C^L x^{\hat{l}} y^m.$$

Since $R_k = -y + Q_k(x, y)$ with $Q_k(0, y) \equiv 0$, the coefficients of Q_k are obtained for $\hat{l} = i_k + 1, \dots, (k+1)d_y + d_x$. We set $l := \hat{l} - i_k$, $m_l := \min \left\{ \left\lfloor \frac{l+i_k}{k+1} \right\rfloor, d_y \right\}$ and we have that:

$$Q_k(x, y) = \sum_{\substack{l=1,\dots,(k+1)d_y+d_x-i_k \\ m=0,\dots,m_l}} b_{l,m} x^l y^m,$$

with:

$$b_{l,m} = \frac{-1}{\omega_0} \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} a_{i,j} \sum_{\substack{|L|=j-m \\ \|L\|=l-i \\ l=i_k+1-i}} \frac{j!}{m! L!} C^L.$$

We are in position to apply the version 2.4 of the Flajolet-Soria formula 2.3 in order to compute the coefficients of $t_k = c_{k+2}x + c_{k+3}x^2 + \dots$. Thus, denoting $Q := (q_{l,m})$ for $l = 1, \dots, (k+1)d_y + d_x - i_k$ and $m = 0, \dots, m_l$, we get that:

$$c_{k+1+p} = \sum_{q=1}^p \frac{1}{q} \sum_{|Q|=q, \|Q\|_1=p, \|Q\|_2=q-1} \frac{q!}{Q!} B^Q.$$

Let us compute:

$$\begin{aligned} b_{l,m}^{q_{l,m}} &= \left(\frac{-1}{\omega_0} \right)^{q_{l,m}} \left(\sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} a_{i,j} \sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m! L!} C^L \right)^{q_{l,m}} \\ &= \left(\frac{-1}{\omega_0} \right)^{q_{l,m}} \sum_{|M_{l,m}|=q_{l,m}, \|M_{l,m}\|_2 \geq m} \frac{q_{l,m}!}{M_{l,m}!} A^{M_{l,m}} \prod_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \left(\sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m! L!} C^L \right)^{m_{i,j}^{l,m}} \\ &\quad \text{where } M_{l,m} = (m_{i,j}^{l,m}) \text{ for } i = 0, \dots, d_x, j = m, \dots, d_y. \end{aligned}$$

For each $m_{i,j}^{l,m}$, we enumerate the terms $\frac{j!}{m! L!} C^L$ with $u = 1, \dots, \alpha_{i,j}$. Subsequently:

$$\begin{aligned} \left(\sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m! L!} C^L \right)^{m_{i,j}^{l,m}} &= \left(\sum_{u=1}^{\alpha_{i,j}} \frac{j!}{m! L_u!} C^{L_u} \right)^{m_{i,j}^{l,m}} \\ &= \sum_{|N_{i,j}|=m_{i,j}^{l,m}} \frac{m_{i,j}^{l,m}!}{N_{i,j}!} \prod_{u=1}^{\alpha_{i,j}} \left(\frac{j!}{m! L_u!} \right)^{n_{i,j,u}^{l,m}} C^{\sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} L_u}, \end{aligned}$$

where $N_{i,j}^{l,m} = (n_{i,j,u}^{l,m})_{u=1,\dots,\alpha_{i,j}}$. Denoting $U_{l,m} := \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} L_u$, one has that:

$$\begin{aligned} |U_{l,m}| &= \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} |L_u| \\ &= \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \left(\sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} \right) (j - m) \\ &= \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} m_{i,j}^{l,m} (j - m) \\ &= \|M_{l,m}\|_2 - m q_{l,m}. \end{aligned}$$

Likewise, one has that:

$$\begin{aligned} \|U_{l,m}\| &= \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} \|L_u\| \\ &= \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \left(\sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} \right) (l + i_k - m(k+1) - i) \\ &= \sum_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} m_{i,j}^{l,m} (l + i_k - m(k+1) - i) \\ &= q_{l,m} [l + i_k - m(k+1)] - \|M_{l,m}\|_1. \end{aligned}$$

So we get that:

$$b_{l,m}^{q_{l,m}} = \left(\frac{-1}{\omega_0}\right)^{q_{l,m}} \sum_{|M_{l,m}|=q_{l,m}, \|M_{l,m}\|_2 \geq m} A^{M_{l,m}} \sum_{\substack{|U_{l,m}|=\|M_{l,m}\|_2-m} \\ \|U_{l,m}\|=q_{l,m}[l+i_k-m(k+1)]-\|M_{l,m}\|_1}} d_{U_{l,m}} C^{U_{l,m}}$$

with $d_{U_{l,m}} := \sum_{\substack{N_{i,j}^{l,m} \\ i=0,\dots,d_x \\ j=m,\dots,d_y}} \frac{q_{l,m}!}{\prod N_{i,j}^{l,m}!} \prod_{i=0,\dots,d_x} \prod_{j=m,\dots,d_y} \left(\frac{j!}{m! L_u!}\right)^{n_{i,j,u}^{l,m}}$,

where the sum is taken over $\left\{ \left(N_{i,j}^{l,m} \right)_{\substack{i=0,\dots,d_x \\ j=m,\dots,d_y}} \text{ such that } |N_{i,j}^{l,m}| = m_{i,j}^{l,m} \text{ and } \sum_{i=0,\dots,d_x} \sum_{j=m,\dots,d_y}^{\alpha_{i,j}} n_{i,j,u}^{l,m} L_u = U_{l,m} \right\}$.

We deduce that:

$$\begin{aligned} B^Q &= \prod_{l=1,\dots,(k+1)d_y+d_x-i_k, m=0,\dots,m_l} b_{l,m}^{q_{l,m}} \\ &= \left(\frac{-1}{\omega_0}\right)^q \prod_{l,m} \left[\sum_{|M_{l,m}|=q_{l,m}, \|M_{l,m}\|_2 \geq m} A^{M_{l,m}} \sum_{\substack{|U_{l,m}|=\|M_{l,m}\|_2-m} \\ \|U_{l,m}\|=q_{l,m}[l+i_k-m(k+1)]-\|M_{l,m}\|_1}} d_{U_{l,m}} C^{U_{l,m}} \right]. \end{aligned}$$

We set $S := \sum_{l,m} M_{l,m}$. So $|S| = \sum_{l,m} q_{l,m} = q$ and $\|S\|_2 \geq \sum_{l,m} m q_{l,m} = \|Q\|_2 = q - 1$.

Moreover, for any fixed S , we set $T_S := \sum_{l,m} U_{l,m}$. So $|T_S| = \sum_{l,m} \|M_{l,m}\|_2 - m q_{l,m} = \|S\|_2 - \|Q\|_2 = \|S\|_2 - q + 1$, and:

$$\begin{aligned} \|T_S\| &= \sum_{l,m} q_{l,m} [l + i_k - m(k+1)] - \|M_{l,m}\|_1 \\ &= \|Q\|_1 + |Q| i_k - \|Q\|_2 (k+1) - \|S\|_1 \\ &= p + q i_k - (q-1)(k+1) - \|S\|_1. \end{aligned}$$

Thus, as desired:

$$\sum_{|Q|=q, \|Q\|_1=p, \|Q\|_2=q-1} \frac{q!}{Q!} B^Q = \left(\frac{-1}{\omega_0}\right)^q \sum_{|S|=q, \|S\|_2 \geq q-1} A^S \sum_{\substack{|T_S|=\|S\|_2-q+1 \\ \|T_S\|=p+q i_k-(q-1)(k+1)-\|S\|_1}} e_{T_S} C^{T_S},$$

where $e_{T_S} := \sum_{\substack{N_{i,j}^{l,m} \\ l,m}} \frac{q!}{\prod_{i,j} N_{i,j}^{l,m}!} \prod_{l,m} \prod_{i,j} \prod_u \left(\frac{j!}{m! L_u!}\right)^{n_{i,j,u}^{l,m}}$ and where the sum is taken over

$$\left\{ \left(N_{i,j}^{l,m} \right)_{\substack{l=1,\dots,(k+1)d_y+d_x-i_k, m=0,\dots,m_l \\ i=0,\dots,d_x, j=m,\dots,d_y}} \text{ such that } \sum_{l,m} \sum_{i,j} |N_{i,j}^{l,m}| = q \text{ and } \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}} n_{i,j,u}^{l,m} L_u = T_S \right\}.$$

□

Remark 4.7. We have seen in the Theorem 4.4 and its proof that $\omega_0 = \pi'_{k_0, i_{k_0}}(c_{k_0+1})$ is the coefficient of the monomial $x^{i_{k_0}+1}y$ in the expansion of $P_{k_0+1}(x, y) = P(x, c_1x + \dots + c_{k_0+1}x^{k_0+1} + x^{k_0+2}y)$, and that $c_{k_0+2} = \frac{-\pi_{k_0, i_{k_0}+1}(c_{k_0+1})}{\omega_0}$ where $\pi_{k_0, i_{k_0}+1}(c_{k_0+1})$ is the coefficient of $x^{i_{k_0}+1}$ in the expansion of $P_{k_0+1}(x, y)$. Expanding $P_{k_0+1}(x, y)$, having done the whole

computations, we deduce that:

$$\begin{cases} \omega_0 &= \sum_{i=0,\dots,d_x, j=0,\dots,d_y} \sum_{|L|=j, \|L\|=i_{k_0}+1-i} \frac{j!}{L!} a_{i,j} C^L; \\ c_{k_0+2} &= \frac{-1}{\omega_0} \sum_{i=0,\dots,d_x, l=0,\dots,d_y-1} \sum_{|L|=l, \|L\|=i_{k_0}+k_0-i-1} \frac{(l+1)!}{L!} a_{i,l+1} C^L. \end{cases}$$

Example 4.8. In order to illustrate the Corollary 4.6 and its proof, we resume the polynomial of the Example 3.4:

$$\begin{aligned} P(x, y) &= a_{0,2}y^2 + (a_{2,0} + a_{2,1}y + a_{2,2}y^2)x^2 \\ P_0(x, y) &= (a_{2,0} + a_{0,2}y^2)x^2 + a_{2,1}yx^3 + a_{2,2}y^2x^4 \\ P_1(x, y) &= (2a_{0,2}c_1y + a_{2,1}c_1)x^3 + (a_{0,2}y^2 + a_{2,1}y + a_{2,2}c_1^2)x^4 + 2a_{2,2}c_1yx^5 + a_{2,2}y^2x^6 \\ &\quad \text{with } a_{2,0} + a_{0,2}c_1^2 = 0 \Leftrightarrow c_1 = \pm \sqrt{\frac{-a_{2,0}}{a_{0,2}}}. \end{aligned}$$

Thus, $i_0 = 2$, $i_1 = 3 = i_0 + 1$, so $k_0 = 0$, $\omega_0 = 2a_{0,2}c_1$. The coefficient c_2 must verify $2a_{0,2}c_1c_2 + a_{2,1}c_1 = 0 \Leftrightarrow c_2 = \frac{-a_{2,1}}{2a_{0,2}}$. We obtain that:

$$\begin{aligned} \omega_0 R_1 &= \omega_0 y + (a_{2,2}c_1^2 + a_{2,1}c_2 + a_{0,2}c_2^2 + (a_{2,1} + 2a_{0,2}c_2)y + a_{0,2}y^2)x + \\ &\quad (2a_{2,2}c_1c_2 + 2a_{2,2}c_1y)x^2 + (a_{2,2}c_2^2 + 2a_{2,2}c_2y + a_{2,2}y^2)x^3. \end{aligned}$$

So the coefficients of the corresponding reduced Henselian equation $y = Q_1(x, y)$ are:

$$\begin{aligned} b_{1,0} &= -(a_{2,2}c_1^2 + a_{2,1}c_2 + a_{0,2}c_2^2)/\omega_0, \quad b_{1,1} = -(a_{2,1} + 2a_{0,2}c_2)/\omega_0 = 0, \\ b_{1,2} &= -a_{0,2}/\omega_0, \quad b_{2,0} = -2a_{2,2}c_1c_2/\omega_0, \quad b_{2,1} = -2a_{2,2}c_1/\omega_0, \\ b_{3,0} &= -a_{2,2}c_2^2/\omega_0, \quad b_{3,1} = -2a_{2,2}c_2/\omega_0, \quad b_{3,2} = -a_{2,2}/\omega_0, \end{aligned}$$

But, by the version 2.4 of the Flajolet-Soria formula 2.3, one has that:

$$\begin{aligned} c_3 &= b_{1,0} = \frac{-a_{2,2}c_1^2 - a_{2,1}c_2 - a_{0,2}c_2^2}{2a_{0,2}c_1}; \\ c_4 &= b_{2,0} + b_{1,0}b_{1,1} = b_{2,0} = \frac{-2a_{2,2}c_1c_2}{2a_{0,2}c_1}; \\ c_5 &= b_{3,0} + b_{1,0}b_{2,1} + b_{1,0}^2b_{1,1} + b_{2,0}b_{1,1} = b_{3,0} + b_{1,0}b_{2,1} + b_{1,0}^2b_{1,2} \\ &= \frac{-a_{2,2}c_2^2}{2a_{0,2}c_1} + \frac{2a_{2,1}a_{2,2}c_1c_2 + 2a_{0,2}a_{2,2}c_1c_2^2 + 2a_{2,2}^2c_1^3}{(2a_{0,2}c_1)^2} - \\ &\quad \frac{a_{0,2}a_{2,1}^2c_2^2 + 2a_{0,2}^2a_{2,1}c_2^3 + 2a_{0,2}a_{2,1}a_{2,2}c_1^2c_2 + a_{0,2}^3c_2^4 + 2a_{0,2}^2a_{2,2}c_1^2c_2^2 + a_{0,2}a_{2,2}^2c_1^4}{(2a_{0,2}c_1)^3}; \\ &\vdots \end{aligned}$$

Remark 4.9. Classically, a series $y_0 = \sum_{n \geq 0} c_n x^n \in K[[x]]$ is algebraic if and only if its coefficients c_n are the diagonal coefficients of the power series expansion of a bivariate rational fraction [Fur67, DL87]. In particular, in the reduced Henselian case $y = Q(x, y)$ (see 2.2), the rational fraction can be written:

$$y_0 = \text{Diag} \left(\frac{y^2 - y^2 \frac{\partial Q}{\partial y}(xy, y)}{y - Q(xy, y)} \right).$$

With the computations in the proof of the Corollary 4.6, we can deduce in the general case $P(x, y) = 0$ a formula for the rational fraction having the c_n as diagonal coefficients of its expansion.

As a consequence of the Theorem 3.5 and Corollary 4.6, we get the following result:

Corollary 4.10. *Let $d_x, d_y \in \mathbb{N}^*$. We set $\mu := 2d_x d_y + 2$ and $M := \frac{1}{2}d_y(d_y + 1)(d_x + 1) + d_x + d_y$. There exists a finite set Λ and for any $\lambda \in \Lambda$, there exist a polynomial $\Omega^{(\lambda)}(x_1, \dots, x_\mu) \in \mathbb{Z}[x_1, \dots, x_\mu]$, $\deg \Omega^{(\lambda)} \leq M$, and for every $p \in \mathbb{N}^*$, a polynomial $\Psi_p^{(\lambda)}(x_1, \dots, x_{\mu+1}) \in \mathbb{Z}[x_1, \dots, x_{\mu+1}]$, $\deg \Psi_p^{(\lambda)} \leq pM$, such that for every $y_0 = \sum_{n \geq 1} c_n x^n$, $c_1 \neq 0$, algebraic with vanishing polynomial of degrees bounded by d_x in x and d_y in y , there exists $\lambda \in \Lambda$ such that for every $p \in \mathbb{N}^*$:*

$$c_{\mu+1+p} = \frac{\Psi_p^{(\lambda)}(c_1, \dots, c_{\mu+1})}{\Omega^{(\lambda)}(c_1, \dots, c_\mu)^p}.$$

Proof. Let $y_0 = \sum_{n \geq 1} c_n x^n$, $c_1 \neq 0$, be algebraic with vanishing polynomial of degree bounded by d_x in x and d_y in y . According to the Theorem 3.5, there is a finite set Λ and for every $\lambda \in \Lambda$, polynomials $a_{i,j}^{(\lambda)}(x_1, \dots, x_N) \in \mathbb{Z}[x_1, \dots, x_N]$ such that:

$$(7) \quad P^{(\lambda)} = \sum_{i \leq d_x, j \leq d_y} a_{i,j}^{(\lambda)}(c_1, \dots, c_N) x^i y^j$$

is a vanishing polynomial for y_0 for a certain $\lambda \in \Lambda$. Enlarging the finite set Λ by indices corresponding to the various $\frac{\partial^k P}{\partial y^k}$, $k = 1, \dots, d_y - 1$, we can assume that there is λ such that y_0 is a simple root of $P^{(\lambda)}$. So the coefficients of y_0 can be computed as in the Corollary 4.6. More precisely, for any $p \in \mathbb{N}^*$:

$$c_{\mu+1+p} = \sum_{q=1}^p \sum_{S \in I_q} \sum_{T_S \in J_S} \frac{m_{S,T_S}}{\omega_0^p} \omega_0^{p-q} A^S C^{T_S}$$

where $I_q = \{(s_{i,j}) \mid |S| = q, \|S\|_2 \geq q - 1\}$,

$$J_S = \{(t_{S,i}) \mid |T_S| = \|S\|_2 - q + 1, \|T_S\| = p + q i_\mu - (q - 1)(\mu + 1) - \|S\|_1\}$$

and $m_{S,T_S} \in \mathbb{Z}$. Note that $C = (c_1, \dots, c_{\mu+1})$ and $A = (a_{i,j})$. It suffices to bound the degrees of the numerator and denominator in the terms of (7). By the Theorem 3.5, $\deg a_{i,j}^{(\lambda)} \leq M - d_y$. So by the Remark 4.7, we deduce that $\deg \omega_0 \leq M$. The degree $d_{q,S}$ of a term $\omega_0^{p-q} A^S C^{T_S}$ is bounded by:

$$(p - q)(M - d_y) + |S|(M - d_y) + |T_S| = (p - q)(M - d_y) + q(M - d_y) + \|S\|_2 - q + 1.$$

But, $\|S\|_2 \leq q d_y$ and $1 \leq q \leq p$. So we get that:

$$d_{q,S} \leq p(M - d_y) + q d_y - q + 1 \leq pM$$

□

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E-mail address: michel.hickel@math.u-bordeaux1.fr, mickael.matusinski@math.u-bordeaux1.fr